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A discontinuous finite difference streamline diffusion method for time-dependent hyperbolic problems

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ABSTRACT

In this article, a new finite element method, discontinuous finite difference streamline diffusion method (DFDSD), is constructed and studied for first-order linear hyperbolic problems. This method combines the benefit of the discontinuous Galerkin method and the streamline diffusion finite element method. Two fully discrete DFDSD schemes (Euler DFDSD and Crank–Nicolson (CN) DFDSD) are constructed by making use of the difference discrete method for time variables and the discontinuous streamline diffusion method for space variables. The stability and optimal L^2 norm error estimates are established for the constructed schemes. This method makes contributions to the discontinuous methods. Finally, a numerical example is provided to show the benefit of high efficiency and simple implementation of the schemes.

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1. Introduction

It is well known that solving the first-order linear hyperbolic problems with the Galerkin finite element method yields only sub-optimal L^2 error estimates, and the Galerkin solutions usually exhibit pseudo-numerical oscillation. To improve the computational accuracy and stability, many non-standard finite element methods were put forward one after another. Among the rest, the discontinuous Galerkin finite element method [1–4] (henceforth mentioned as the DG method) and the streamline diffusion finite element method [5–8] (henceforth mentioned as the SD method), are two kinds of algorithms which are successful and have brilliant characteristics. Concretely, the DG method is an upwind-type algorithm. It begins from an inflow boundary, and computes element by element; the computation is simple and can be done locally in a parallel manner. The SD method involves Petrov–Galerkin-type artificial viscosity by the introduction of an artificial viscosity (diffusion) term along the streamline direction, leading to a computation process with good stability. But as the SD method is a kind of implicit method, it needs to solve wholly the discrete equations among the computational domain, and the workload is large. [9] has already combined the DG method with the SD method, and put forward the discontinuous streamline diffusion FEM. We call it DSD for short, hereinafter. Its fundamental idea is: retain the basic structure of DG algorithms, but while computing explicitly from the inflow to the outflow, element by element, change the Galerkin structure to the SD structure. So it retains the upwind, explicit characteristic of the SD method, and improves the stability of SD. Hence [9] treats the time variables and space variables in the same manner when using the SD method to solve time-dependent problems, by adopting the space–time finite element method. This process increases the dimension of solving a problem by one virtually, leading to a lot of difficulties while treating high-dimensional and nonlinear problems. According to the above analysis, in this article, two fully discrete DFDSD methods (Euler DFDSD and CN DFDSD) are constructed by making use of the difference discrete method for time variables and the DSD discrete method for space variables. The stability and convergence of the schemes are analysed and numerical experiments are carried out.

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2. DFDSD schemes for the first-order linear hyperbolic problem

Let $\Omega \in R^2$ be a polygon with boundary Γ and $[0, T]$ be the time interval. Consider the following first-order linear hyperbolic problems:

$$\frac{\partial u}{\partial t} + \beta(x,t)\cdot\nabla u + \sigma(x,t)u = f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{2.1}$$

$$u(x,t) = g(x,t), \quad (x,t) \in \Gamma_-(t) \times (0,T), \tag{2.2}$$

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{2.3}$$

where $\beta = (\beta_1, \beta_2)$, ∇ denotes the gradient operator, $\Gamma = \Gamma_-(t) \cup \Gamma_+(t)$:

$$\Gamma_-(t) = \{x \in \Gamma : \beta(x,t)\cdot\gamma(x) < 0\}, \quad \gamma(x) \text{ is the outward unit vector of } \Gamma \text{ at } x,$$
$$\Gamma_+(t) = \{x \in \Gamma : \beta(x,t)\cdot\gamma(x) \geq 0\} = \Gamma \setminus \Gamma_-(t).$$

Γ_- is referred to as the inflow boundary at time t of (2.1) and Γ_+ is called the outflow boundary at time t.

Suppose $\beta_1, \beta_2, \sigma \in W^{1,\infty}(\Omega \times [0,T]) \cap C(\bar{\Omega} \times [0,T]), f \in L^\infty(L^2(\Omega)), g \in L^\infty(L^2(\Gamma_-(t))), u_0 \in L^2(\Omega)$.

As in [10], to simplify the theoretical analysis, we assume that the direction of $\beta(x,t)$ is independent of t on the space boundary Γ (especially, when $\beta(x,t)$ is independent of t, it satisfies the condition naturally). Therefore $\Gamma_-(t)$ is a fixed curve, denoted by Γ_-. Applying the finite difference discrete method to t, let $\Delta t = \tau$ be the time step, $t^n = n\tau,\ n = 0, 1, \ldots, N = [T/\tau]$. Consider the quasi-uniform triangular partition for $\bar{\Omega}$: $\mathcal{T}_h = \{k : k \in \bar{\Omega}\}$ with mesh parameter h $(0 < h < h_0 < 1)$, where k is the element of $\mathcal{T}_h$ and ∂k denotes the boundary of k.

Denote by $P_r(k)$ the set of polynomials with degree $\leq r$ on k. Define

$$V_h = \{v \in L^2(\Omega), v|_k \in P_r(k), \forall k \in \mathcal{T}_h\}, \quad r \geq 0.$$

Define $\beta^n(x) = \beta(x, t^n)$, for $\forall k \in \mathcal{T}_h$, and suppose that ∂k consists of straight line sides l_j $(j = 1, 2, 3)$. Denote by $\gamma(x)$ the outward unit vector of ∂k. On time level $t = t^n$, for $\forall k \in \mathcal{T}_h$, define

$$\bar{\beta}_j^n = \frac{1}{|l_j|}\int_{l_j} \beta^n(x)\mathrm{d}s, \quad j = 1,2,3 \ (|l_j| \text{ is the length of } l_j),$$
$$\bar{\beta}^n(x) = \bar{\beta}_j^n, \quad \text{for } \forall x \in l_j,\ j = 1,2,3,$$
$$\partial k_-^n = \{x \in \partial k, \bar{\beta}^n(x)\cdot\gamma(x) < 0\}, \quad \partial k_+^n = \partial k \setminus \partial k_-^n.$$

∂k_-^n and ∂k_+^n are called the inflow and the outflow boundary of element k, respectively.

Note that when $v \in V_h$, $v|_{\partial k}$ may be discontinuous, on $t = t^n$, for $v, w \in V_h$ and $x \in \partial k$, define

$$v_+^n(x) = \lim_{s\to 0^+} v(x + s\bar{\beta}^n(x)), \qquad v_-^n(x) = \lim_{s\to 0^-} v(x + s\bar{\beta}^n(x)), \qquad [v^n(x)] = v_+^n(x) - v_-^n(x),$$
$$\langle v, w\rangle_{\partial k_-^n} = \int_{\partial k_-^n} vw|\bar{\beta}^n\cdot\gamma|\mathrm{d}s, \qquad |v|^2_{\partial k_-^n} = \langle v, w\rangle_{\partial k_-^n},$$
$$\langle v, w\rangle_{\Gamma_-^n} = \sum_{\partial k_-^n \subset \Gamma_-} vw\langle v, w\rangle_{\partial k_-^n}, \qquad |v|^2_{\Gamma_-^n} = \langle v, w\rangle_{\Gamma_-^n}.$$

Likewise, $\langle v, w\rangle_{\partial k_+^n}$, $|v|_{\partial k_+^n}$, $\langle v, w\rangle_{\Gamma_+^n}$, $|v|_{\Gamma_+^n}$ can also be defined. Denote

$$(v,w)_k = \int_k vw\mathrm{d}x, \qquad \|v\|_k = (v,v)_k, \qquad (v,w) = \int_\Omega vw\mathrm{d}x, \qquad \|v\| = (v,v).$$

2.1. The Euler DFDSD scheme

Let $v^n(x) = v(x, t^n),\ \Delta_t v^n = (v^n - v^{n-1})/\tau,\ v_\beta = \beta(x)\cdot\nabla v$. Then, at time $t = t^n$, problem (2.1)–(2.3) can be written as

$$\Delta_t u^n + u^n_{\beta^n} + \sigma^n u^n = f^n + E_1^n, \quad n = 1, 2, \ldots, N, \tag{2.4}$$

$$u_-^n|_{\Gamma_-} = g^n, \tag{2.5}$$

$$u^0 = u_0, \quad x \in \Omega \tag{2.6}$$

where $E_1^n = \Delta_t u^n - \left(\frac{\partial u}{\partial t}\right)^n$ is the truncation error.

Omitting E_1^n from (2.4) and noting the definition of the DSD scheme [9], the Euler DFDSD scheme of (2.1)–(2.3) is defined as follows: find $U^n \in V_h,\ n = 0, 1, 2, \ldots, N$, such that $\forall k \in \mathcal{T}_h$,

$$(\Delta_t U^n + U^n_{\beta^n} + \sigma^n U^n, v + \delta v_{\beta^n})_k + \langle \tilde{\sigma}^n [U^n], v_+\rangle_{\partial k_-} = (f^n, v + \delta v_{\beta^n})_k, \quad \forall v \in P_r(k), \tag{2.7}$$

$$U^n_-|_{\partial k_-} = g^n, \quad \text{when } \partial k_- \in \Gamma_-, \tag{2.8}$$

$$(U^0 - u_0, v)_k = 0, \quad \forall v \in P_r(k) \tag{2.9}$$

where $\tilde{\sigma}^n = 1 + \delta\sigma^n$. $\tau = \bar{C}h$, $\bar{C} > 0$ can be chosen arbitrarily. $\delta = \bar{\bar{C}}h$, $0 < \bar{\bar{C}} < \bar{C}/4$.

Summing (2.7)–(2.9) over $k \in \mathcal{T}_h$, we have the whole form of (2.7)–(2.9)

$$(\Delta_t U^n + U^n_{\beta^n} + \sigma^n U^n, v + \delta v_{\beta^n})_\Omega + \sum_{k \in \mathcal{T}_h^n} \langle \tilde{\sigma}^n [U^n], v_+\rangle_{\partial k_-} = (f^n, v + \delta v_{\beta^n})_\Omega, \quad \forall v \in V_h, \tag{2.10}$$

$$U^n_-|_{\Gamma_-} = g^n, \tag{2.11}$$

$$(U^0 - u_0, v) = 0, \quad \forall v \in V_h. \tag{2.12}$$

2.2. The Crank–Nicolson DFDSD scheme

Let $t_n = (t^{n-1} + t^n)$, $v_n = v(x, t_n)$, $\tilde{v}^n(x) = (v^n(x) + v^{n-1}(x))/2$. Then on level $t = t_n$, (2.1) can be written as

$$\Delta_t u^n + \tilde{u}^n_{\beta_n} + \sigma_n \tilde{u}^n = f_n + E_2^n, \quad n = 1, 2, \ldots, N, \tag{2.13}$$

where the truncation error is

$$E_2^n = \Delta_t u^n - \left(\frac{\partial u}{\partial t}\right)_n + \beta_n \cdot \nabla(u_n - \tilde{u}^n) + \sigma(u_n - \tilde{u}^n). \tag{2.14}$$

Omitting E_2^n from (2.13), the CN DFDSD scheme is defined as follows: find $U^n : \{t^n\}_{n=1}^N \to V_h$, such that for $\forall k \in \mathcal{T}_h$,

$$(\Delta_t U^n + \tilde{U}^n_{\beta_n} + \sigma_n \tilde{U}^n, v + \delta v_{\beta_n})_k + \langle \tilde{\sigma}_n [\tilde{U}^n], v_+\rangle_{\partial k_-} = (f_n, v + \delta v_{\beta_n})_k, \quad \forall v \in P_r(k), \tag{2.15}$$

$$U^n_-|_{\partial k_-} = g^n, \quad \text{when } \partial k_- \in \Gamma_-, \tag{2.16}$$

$$(U^0 - u_0, v)_k = 0, \quad \forall v \in P_r(k) \tag{2.17}$$

where $\tilde{\sigma}_n = 1 + \delta\sigma_n$. $\tau = C''\sqrt{h}$, $C'' > 0$ can be chosen arbitrarily. $\delta = C'h$, $0 < C' < 2C''^2$.

Summing (2.15)–(2.17) over $k \in \mathcal{T}_h$, we have the whole form of (2.15)–(2.17)

$$(\Delta_t U^n + \tilde{U}^n_{\beta_n} + \sigma_n \tilde{U}^n, v + \delta v_{\beta_n})_\Omega + \sum_{k \in \mathcal{T}_h^n} \langle \tilde{\sigma}_n [\tilde{U}^n], v_+\rangle_{\partial k_-} = (f_n, v + \delta v_{\beta_n})_\Omega, \quad \forall v \in V_h, \tag{2.18}$$

$$U^n_-|_{\Gamma_-} = g^n, \tag{2.19}$$

$$(U^0 - u_0, v) = 0, \quad \forall v \in V_h. \tag{2.20}$$

3. Analysis for the Euler DFDSD scheme

3.1. Stability analysis for the Euler DFDSD scheme

For simplicity in notations, denote $\sum \triangleq \sum_{k \in \mathcal{T}_h}$, $\bigcup \triangleq \bigcup_{k \in \mathcal{T}_h}$. On $t = t^n$, set $Q^n_- = \cup \partial k^n_-$, $Q^n_+ = \cup \partial k^n_+$, and denote $\langle v, w\rangle_{Q^n_-} = \sum \langle v, w\rangle_{\partial k^n_-}$, $\langle v, w\rangle_{Q^n_+} = \sum \langle v, w\rangle_{\partial k^n_+}$,

$$B(w^n, v; w^{n-1}) \triangleq \sum (\Delta_t w^n + w^n_{\beta^n} + \sigma^n w^n, v + \delta v_\beta)_k + \langle \tilde{\sigma}^n [w^n], v_+\rangle_{Q^n_-}. \tag{3.1}$$

Lemma 3.1. *There exist constants $C^*, C^{**} > 0$, independent of k, h, n, such that for $\forall v \in P_r(k)$,*

$$\|v\|_{L^2(\partial k)} \le C^* h^{-\frac{1}{2}} \|v\|_k, \quad \forall k \in \mathcal{T}_h, \tag{3.2}$$

$$\left| \int_{\partial k} v^2 (\beta^n - \bar{\beta}^n) \cdot \gamma \, ds \right| \le C^{**} \|v\|_k^2, \quad \forall k \in \mathcal{T}_h. \tag{3.3}$$

Proof. Estimate (3.2) can be derived from the quasi-uniformity of $\mathcal{T}_h$ and the inverse estimation $\|v\|_{L^\infty(k)} \le M_0 h^{-1} \|v\|_k$ for $v \in P_r(k)$. The inequality (3.3) follows from (3.2) and the fact $\|\beta^n - \bar{\beta}^n\|_{L^\infty(\partial k)} \le M_1 h \|\beta\|_{L^\infty(C^1(\bar{\Omega}))}$. $\square$

Lemma 3.2. *There exist constants $C_0, C_1 > 0$ independent of k, h, n, such that for $\forall w^n, w^{n-1} \in V_h$ and $\forall w_-^n|_{\Gamma_-} \in L^2(\Gamma_-)$,*

$$B(w^n, w^n; w^{n-1}) + C_0\|w^n\|^2 + \frac{\sigma_1}{2}|w_-^n|_{\Gamma_-}^2 \geq \frac{1}{2}\left[\Delta_t\|w^n\|^2 + \sigma_0|[w^n]|_{Q_-^n}^2 + \sigma_0|w_-^n|_{\Gamma_+}^2\right] + C_1\delta\|w_{\beta^n}^n\|^2 + \frac{1}{4}\|\Delta_t w^n\|^2, \tag{3.4}$$

where

$$\Delta_t\|w^n\|^2 \triangleq (\|w^n\|^2 - \|w^{n-1}\|^2)/\tau, \qquad \sigma_0 = \inf_{x,t}|\tilde{\sigma}(x,t)|, \qquad \sigma_1 = \sup_{x,t}|\tilde{\sigma}(x,t)|.$$

Proof. By definition (3.1),

$$B(w^n, w^n; w^{n-1}) = \sum\left(\Delta_t w^n + w_{\beta^n}^n + \sigma^n w^n, w^n + \delta w_\beta^n\right)_k + \langle\tilde{\sigma}^n[w^n], w_+^n\rangle_{Q_-^n}. \tag{3.5}$$

It is easy to note that

$$(\Delta_t w^n, w^n) = \frac{1}{2}\left(\tau\|\Delta_t w^n\|^2 + \Delta_t\|w^n\|^2\right), \qquad \left(w_{\beta^n}^n, \delta w_{\beta^n}^n\right) = \delta\|w_{\beta^n}^n\|^2,$$

$$\left|(\Delta_t w^n, \delta w_{\beta^n}^n)\right| \leq \frac{\tau}{4}\|\Delta_t w^n\|^2 + \frac{\delta^2}{\tau}\delta\|w_{\beta^n}^n\|^2,$$

$$\left(w_{\beta^n}^n + \sigma^n w^n, w^n\right)_k + \left(\sigma^n w^n, \delta w_{\beta^n}^n\right)_k = \left(\left(\sigma^n - \frac{1}{2}\mathrm{div}\,\beta^n\right)w^n, w^n\right)_k - \frac{\delta}{2}((\sigma_{\beta^n}^n + \sigma^n\mathrm{div}\,\beta^n)w^n, w^n)_k + \frac{1}{2}\int_{\partial k}\tilde{\sigma}^n(w^n)^2\beta^n\cdot\gamma\,\mathrm{d}s,$$

$$\int_{\partial k}\tilde{\sigma}^n(w^n)^2\beta^n\cdot\gamma\,\mathrm{d}s = \int_{\partial k}\tilde{\sigma}^n(w^n)^2\bar{\beta}^n\cdot\gamma\,\mathrm{d}s + \int_{\partial k}\tilde{\sigma}^n(w^n)^2(\beta^n - \bar{\beta}^n)\cdot\gamma\,\mathrm{d}s,$$

$$\int_{\partial k}\tilde{\sigma}^n(w^n)^2\bar{\beta}^n\cdot\gamma\,\mathrm{d}s = \int_{\partial k_+^n}\tilde{\sigma}^n(w_-^n)^2\bar{\beta}^n\cdot\gamma\,\mathrm{d}s - \int_{\partial k_-^n}\tilde{\sigma}^n(w_+^n)^2|\bar{\beta}^n\cdot\gamma|\,\mathrm{d}s.$$

Substituting these expressions into (3.5), and taking $C_1 \leq 1 - \frac{\bar{\bar{C}}}{\bar{C}}$, then

$$\begin{aligned}B(w^n, w^n; w^{n-1}) \geq{}& \frac{1}{2}\Delta_t\|w^n\|^2 + C_1\delta\|w_{\beta^n}^n\|^2 - \left\|\sigma^n - \frac{1}{2}\mathrm{div}\,\beta^n\right\|_{L^\infty(L^\infty(\Omega))}\|w^n\|^2\\ &- \frac{\delta}{2}\|\sigma_{\beta^n}^n - \sigma^n\mathrm{div}\,\beta^n\|_{L^\infty(L^\infty(\Omega))}\|w^n\|^2 + \frac{1}{4}\tau\|\Delta_t w^n\|^2 + \frac{1}{2}\langle\tilde{\sigma}^n w_-^n, w_-^n\rangle_{Q_+^n}\\ &- \frac{1}{2}\langle\tilde{\sigma}^n w_+^n, w_+^n\rangle_{Q_-^n} + \langle\tilde{\sigma}^n[w^n], w_+^n\rangle_{Q_-^n} - \frac{1}{2}\sum\left|\int_{\partial k}\tilde{\sigma}^n(w^n)^2(\beta^n - \bar{\beta}^n)\cdot\gamma\,\mathrm{d}s\right|.\end{aligned}$$

Note that

$$\langle\tilde{\sigma}^n w_-^n, w_-^n\rangle_{Q_+^n} = \langle\tilde{\sigma}^n w_-^n, w_-^n\rangle_{Q_-^n} - \langle\tilde{\sigma}^n w_-^n, w_-^n\rangle_{\Gamma_-} + \langle\tilde{\sigma}^n w_-^n, w_-^n\rangle_{\Gamma_+}.$$

Applying (3.3) to $\int_{\partial k}\tilde{\sigma}^n(w^n)^2(\beta^n - \bar{\beta}^n)\cdot\gamma\,\mathrm{d}s$ and taking

$$C_0 = \left\|\sigma^n - \frac{1}{2}\mathrm{div}\,\beta^n\right\|_{L^\infty(L^\infty(\Omega))} + \frac{\delta}{2}\|\sigma_{\beta^n}^n - \sigma\beta^n\mathrm{div}\,\beta^n\|_{L^\infty(L^\infty(\Omega))} + \frac{1}{2}C^{**},$$

(3.4) can be obtained immediately. □

Theorem 3.1. *For $\Delta t\ (=\tau)$ sufficiently small, the Euler DFDSD scheme (2.10)–(2.12) has a unique solution $\{U^n\}_{n=1}^N$, and the following stability estimate holds:*

$$\max_{1\leq n\leq N}\|U^n\|^2 + \sum_{n=1}^N\left(|[U^n]|_{Q_-^n}^2 + |U_-^n|_{\Gamma_+}^2\right)\tau + \sum_{n=1}^N\left(\tau\|\Delta_t U^n\|^2 + \delta\|U_{\beta^n}^n\|^2\right)\tau \leq C(\|f\|_{L^\infty(L^2(\Omega))}^2 + \|g\|_{L^\infty(L^2(\Gamma_-))}^2 + \|u_0\|^2), \tag{3.6}$$

where C is independent of τ, h.

Proof. From (2.10)–(2.12), we have

$$B(U^n, U^n; U^{n-1}) = (f^n, U^n + \delta U^n_{\beta^n}), \quad n = 1, 2, \ldots, N. \tag{3.7}$$

By the ε inequality

$$(f^n, U^n + \delta U^n_{\beta^n}) \le \left(\frac{1}{4} + \frac{\tau}{4}\right) \|f^n\|^2 + \|U^n\|^2 + \frac{\delta^2}{\tau} \|U^n_{\beta^n}\|^2. \tag{3.8}$$

Again from Lemma 3.2 and (2.11),

$$\begin{aligned} &\Delta_t \|U^n\|^2 + 2\sigma_0 |[U^n]|^2_{Q^n_-} + 2\sigma_0 |U^n_-|^2_{\Gamma_+} + \left(1 - \frac{2\delta}{\tau}\right) \delta \|U^n_{\beta^n}\|^2 + \frac{\tau}{2} \|\Delta_t U^n\|^2 \\ &\quad \le 2(1 + C_0)\|U^n\|^2 + 2\|f^n\|^2 + 2\sigma_1 |g^n|^2, \quad n = 1, 2, \ldots, N. \end{aligned} \tag{3.9}$$

Multiplying by τ for the above inequality and summing from 1 to n, then using the Gronwall inequality and noting that $\|U^0\| \le \|u_0\|$, we get: if τ is small enough such that $1 - 2(1 + C_0)\tau \ge \mu_0 > 0$, and

$$\sum_{n=1}^{N} \|f^n\|^2 \tau \le T \|f\|^2_{L^\infty(L^2(\Omega))}, \qquad \sum_{n=1}^{N} |g^n|^2 \tau \le T \|f\|^2_{L^\infty(L^2(\Gamma_-))}, \tag{3.10}$$

then

$$\begin{aligned} &\|U^n\|^2 + \sum_{n=1}^{N} \left(|[U^n]|^2_{Q^n_-} + |U^n_-|^2_{\Gamma_+} \right) \tau + \sum_{n=1}^{N} \left(\tau \|\Delta_t U^n\|^2 + \delta \|U^n_{\beta^n}\|^2 \right) \tau \\ &\quad \le C(\|f\|^2_{L^\infty(L^2(\Omega))} + \|g\|^2_{L^\infty(L^2(\Gamma_-))} + \|u_0\|^2), \quad n = 1, 2, \ldots, N. \end{aligned} \tag{3.11}$$

The conclusion (3.6) is proved. □

3.2. Error estimates for the Euler DFDSD scheme

Let u be the solution of (2.1)–(2.3). Assume that

$$u \in L^\infty(H^{r+1}(\Omega)) \cap C(\bar{\Omega} \times [0, T]), \qquad \frac{\partial u}{\partial t} \in L^2(H^{r+1}(\Omega)), \qquad \frac{\partial^2 u}{\partial t^2} \in L^2(L^2(\Omega)). \tag{3.12}$$

Then the truncation error E^n_1 in the following equation

$$B(u^n, v; u^{n-1}) = (f^n + E^n_1, v + \delta v^n_{\beta^n}), \quad \forall v \in V_h,\ n = 1, 2, \ldots, N \tag{3.13}$$

can be bounded by

$$\|E^n_1\|^2 \le K_1 \tau \left\| \frac{\partial^2 u}{\partial t^2} \right\|^2_{L^2(t^{n-1}, t^n; L^2(\Omega))} \tag{3.14}$$

where K_1 is independent of $h\tau$. Note that $[u^n] = 0$; then from (3.13) and (2.10)–(2.12)

$$B(u^n - U^n, v; u^{n-1} - U^{n-1}) = E^n_1, v + \delta v_{\beta^n}, \quad \forall v \in V_h,\ n = 1, 2, \ldots, N, \tag{3.15}$$

$$(u^n_- - U^n_-)|_{\Gamma_-} = 0, \tag{3.16}$$

$$(u^0 - U^0, v) = 0, \quad \forall v \in V_h. \tag{3.17}$$

Define $\tilde{u}(t) : [0, T] \to V_h$, such that $\forall k \in \mathcal{T}_h$,

$$(\tilde{u}(t) - u(t), v) = 0, \quad \forall v \in P_r(k),\ t \in [0, T]. \tag{3.18}$$

Denote $\xi^n = U^n - u^n$, $\eta^n = u^n - \tilde{u}^n$, $e^n = u^n - U^n = \eta^n - \xi^n$, and take $\tilde{u}^n_-|_{\Gamma_-} = g^n$; then

$$B(\xi^n, v; \xi^{n-1}) = B(\eta^n, v; \eta^{n-1}) - (E^n_1, \xi^n + \delta \xi^n_{\beta^n}), \quad \forall v \in V_h,\ n = 1, 2, \ldots, N, \tag{3.19}$$

$$\xi^n_-|_{\Gamma_-} = 0, \qquad \eta^n_-|_{\Gamma_-} = 0, \tag{3.20}$$

$$\xi^0 = 0. \tag{3.21}$$

Applying Lemma 3.2 and boundary condition (3.20), we can get

$$\begin{aligned}&\frac{1}{2}\left[\Delta_t\|\xi^n\|^2+\sigma_0|[\xi^n]|^2_{Q^n_-}+\sigma_0|\xi^n_-|^2_{\Gamma_+}\right]+C_1\delta\|\xi^n_{\beta^n}\|^2+\frac{\tau}{4}\|\Delta_t\xi^n\|^2\\&\quad\le B(\xi^n,\xi^n;\eta^{n-1})+C_0\|\xi^n\|^2\\&\quad=B(\eta^n,\xi^n;\eta^{n-1})-(E_1^n,\xi^n+\delta\xi^n_{\beta^n})+C_0\|\xi^n\|^2\\&\quad\le B(\eta^n,\xi^n;\eta^{n-1})+\bar{C}_0(\varepsilon)\|\xi^n\|^2+C_2\|E_1^n\|^2+\varepsilon\delta\|\xi^n_{\beta^n}\|^2.\end{aligned}\tag{3.22}$$

Lemma 3.3. *There exists a constant $C>0$, such that*

$$\begin{aligned}\Delta_t\|\xi^n\|^2+|[\xi^n]|^2_{Q^n_-}+|\xi^n_-|^2_{\Gamma_+}+\delta\|\xi^n_{\beta^n}\|^2+\tau\|\Delta_t\xi^n\|^2&\le C\{\|\xi^n\|^2+\|\eta^n\|^2+h\|\eta^n\|_1^2\\&\quad+|\eta^n|^2_{Q^n_-}+|\eta^n_-|^2_{\Gamma_+}+\|\Delta_t\eta^n\|^2+\|E_1^n\|^2\}.\end{aligned}\tag{3.23}$$

Proof. In fact

$$B(\eta^n,\xi^n;\eta^{n-1})=\sum(\Delta_t\eta^n+\eta^n_{\beta^n}+\sigma^n\eta^n,\xi^n+\delta\xi^n_{\beta^n})_k+\langle\tilde{\sigma}[\eta^n],\xi^n_+\rangle_{Q^n_-}.\tag{3.24}$$

Since $\xi^n|_k\in P_r(k)$, we have from the definition of $\tilde{u}$

$$(\Delta_t\eta^n,\xi^n)_k=\frac{1}{\tau}(\eta^n-\eta^{n-1},\xi^n)_k=0,\quad\forall k\in\mathcal{T}_h.\tag{3.25}$$

Integrating by parts yields

$$\begin{aligned}(\eta^n_{\beta^n}+\sigma^n\eta^n,\xi^n)_k&=-(\eta^n,\xi^n_{\beta^n})_k+((\sigma^n-\operatorname{div}\beta^n)\eta^n,\xi^n)_k+\int_{\partial k}\eta^n\xi^n\beta^n\cdot\gamma\,\mathrm{d}s\\&\le-(\eta^n,\xi^n_{\beta^n})_k+\int_{\partial k}\eta^n\xi^n\beta^n\cdot\gamma\,\mathrm{d}s+\|\sigma^n-\operatorname{div}\beta^n\|_{L^\infty(L^\infty(\Omega))}\|\eta\|_k\cdot\|\xi\|_k.\end{aligned}\tag{3.26}$$

Let O_k be the geometrical center of element $k\in\mathcal{T}_h$; then $\beta^n(O_k)\cdot\nabla\xi^n|_k\in P_r(k)$ and $|\beta^n(x)-\beta^n(O_k)|\le Ch_k$. Using the inverse estimation of $P_r(k)$, we have

$$(\eta^n,\xi^n_{\beta^n})_k=(\eta^n,(\beta^n(x)-\beta^n(O_k))\cdot\nabla\xi^n)_k\le C\|\eta^n\|_k\cdot\|\xi^n\|_k.\tag{3.27}$$

In addition, it is easy to prove

$$\begin{aligned}(\sigma^n\eta^n,\delta\xi^n_{\beta^n})_k&=\int_{\partial k}\delta\sigma^n\eta^n\xi^n\beta^n\cdot\gamma\,\mathrm{d}s-\delta((\sigma^n_{\beta^n}+\sigma^n\operatorname{div}\beta^n)\eta^n,\xi^n)\\&\le\int_{\partial k}\delta\sigma^n\eta^n\xi^n\beta^n\cdot\gamma\,\mathrm{d}s+\delta\|\sigma^n_{\beta^n}+\sigma^n\operatorname{div}\beta^n\|_{L^\infty(L^\infty(\Omega))}\|\eta\|_k\|\xi\|_k\end{aligned}\tag{3.28}$$

$$(\eta^n_{\beta^n},\delta\xi^n_{\beta^n})_k\le C\delta\|\eta^n\|_{1,k}\cdot\|\xi^n_{\beta^n}\|_k\le C\delta\|\eta^n\|^2_{1,k}+\varepsilon\delta\|\xi^n_{\beta^n}\|^2_k.\tag{3.29}$$

Note that

$$\begin{aligned}\int_{\partial k}\eta^n\xi^n\beta^n\cdot\gamma\,\mathrm{d}s+\int_{\partial k}\delta\sigma^n\eta^n\xi^n\beta^n\cdot\gamma\,\mathrm{d}s&=\int_{\partial k}\tilde{\sigma}^n\eta^n\xi^n\beta^n\cdot\gamma\,\mathrm{d}s\\&=\langle\tilde{\sigma}^n\eta^n_-,\xi^n_-\rangle_{\partial k^n_+}-\langle\tilde{\sigma}^n\eta^n_+,\xi^n_+\rangle_{\partial k^n_-}+\int_{\partial k}\tilde{\sigma}^n\eta^n\xi^n(\beta^n-\bar{\beta}^n)\cdot\gamma\,\mathrm{d}s.\end{aligned}\tag{3.30}$$

Applying Lemma 3.2 and the trace inequality, we have

$$\begin{aligned}\int_{\partial k}\tilde{\sigma}^n\eta^n\xi^n(\beta^n-\bar{\beta}^n)\cdot\gamma\,\mathrm{d}s&\le\sigma_1\left(\int_{\partial k}(\eta^n)^2|\beta^n-\bar{\beta}^n|\mathrm{d}s\right)^{\frac12}\cdot\left(\int_{\partial k}(\xi^n)^2|\beta^n-\bar{\beta}^n|\mathrm{d}s\right)^{\frac12}\\&\le Ch^{\frac12}\|\eta^n\|_{L^2(\partial k)}\cdot\|\xi^n\|_k\le Ch^{\frac12}\|\eta^n\|_{H^1(k)}\cdot\|\xi^n\|_k\\&\le\|\xi^n\|_k^2+C^2h\|\eta^n\|^2_{1,k}.\end{aligned}\tag{3.31}$$

Combining (3.24)–(3.31), and noting that (3.20), we obtain

$$\begin{aligned}B(\eta^n,\xi^n;\eta^{n-1})&\le C_3(\|\eta^n\|^2+\|\xi^n\|^2)+\varepsilon\delta\|\xi^n_{\beta^n}\|^2+C_2h\|\eta^n\|_1^2+\langle\tilde{\sigma}^n\eta^n_-,\xi^n_-\rangle_{Q^n_+}-\langle\tilde{\sigma}^n\eta^n_+,\xi^n_+\rangle_{Q^n_-}+\langle\tilde{\sigma}^n[\eta^n],\xi^n_+\rangle_{Q^n_-}\\&\le C_4(\|\eta^n\|^2+\|\xi^n\|^2+h\|\eta^n\|_1^2)+\varepsilon\delta\|\xi^n_{\beta^n}\|^2+\frac{\sigma_0}{4}(|[\xi^n]|^2_{Q^n_-}+\|\xi^n_-\|^2_{\Gamma_+})+\frac{\sigma_1}{\sigma_0}(|\eta^n_-|^2_{Q^n_-}+|\eta^n_-|^2_{\Gamma_+}).\end{aligned}$$

Substituting the above inequality into (3.22), and taking ε small enough, such that $C_1-2\varepsilon>0$, estimate (3.23) is proved. □

Theorem 3.2. *Let u, $\{U^n\}$ be the solutions for problem (2.1)–(2.3) and DFDSD scheme (2.10)–(2.12) respectively. Assume that condition (3.12) holds; then there exists a constant $C > 0$ independent of τ, h, such that for τ small enough*

$$\max_{0\le n\le N} \|e^n\|^2 + \sum_{n=1}^{N}(|[e^n]|^2_{Q^n_-} + |e^n_-|^2_{\Gamma_+})\tau + \sum_{n=1}^{N}(\tau\|\Delta_t e^n\|^2 + \delta\|e^n_{\beta^n}\|^2)\tau \le C(h^{2r+1} + \tau^2) \tag{3.32}$$

where $U^n_+|_{\Gamma_+} = u^n_+|_{\Gamma_+} = \tilde{u}^n_+|_{\Gamma_+} = 0$ are specified.

Proof. Multiplying (3.23) by τ, summing for n, applying the Gronwall inequality, and recalling $\xi^0 = 0$, we obtain for τ small enough

$$\|\xi^n\|^2 + \sum_{j=1}^{n}(|[\xi^j]|^2_{Q^j_-} + |\xi^j_-|^2_{\Gamma_+} + \delta\|\xi^j_{\beta j}\|^2 + \tau\|\Delta_t\xi^j\|^2)\tau$$
$$\le C_5\sum_{j=1}^{n}(\|\eta^j\|^2 + h\|\eta^j\|^2_1 + |\eta^j_-|^2_{Q^j_-} + |\eta^j|^2_{\Gamma_+} + \|\Delta_t\eta^j\|^2 + \|E^j_1\|^2)\tau. \tag{3.33}$$

From (3.14), we have

$$\sum_{j=1}^{n}\|E^j_1\|^2 \le K_1\tau^2\sum_{j=1}^{n}\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L^2(t^{j-1},t^j;L^2(\Omega))} \le K_1\tau^2\sum_{j=1}^{n}\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L^2(L^2(\Omega))}. \tag{3.34}$$

Therefore from (3.33)

$$\max_{0\le n\le N}\|\xi^n\|^2 + \sum_{n=1}^{N}(|[\xi^n]|^2_{Q^n_-} + |\xi^n_-|^2_{\Gamma_+} + \delta\|\xi^n_{\beta^n}\|^2 + \tau\|\Delta_t\xi^n\|^2)\tau$$
$$\le C_6\sum_{n=1}^{N}(\|\eta^n\|^2 + h\|\eta^n\|^2_1 + |\eta^n_-|^2_{Q^j_-} + |\eta^n|^2_{\Gamma_+} + \|\Delta_t\eta^n\|^2 + \tau^2)\tau. \tag{3.35}$$

In addition, from [10], we know that there exists K_i $(i = 2, 3, \ldots, 6)$ independent of τ, h, such that

$$\|\eta^n\| \le K_2h^{r+1}, \qquad \|\eta^n\|_{1,k} \le K_3h^r, \qquad |[\eta^n]|_{Q^n_-} \le K_4h^{r+\frac{1}{2}}, \tag{3.36}$$

$$\|\eta^n_-\|_{\Gamma^n_+} \le K_5h^{r+\frac{1}{2}}, \qquad \sum_{n=1}^{N}\|\Delta_t\eta^n\|^2\tau \le K_6h^{2r+2}. \tag{3.37}$$

Thus from (3.33) and applying the triangle inequality, the convergence order estimate (3.32) is obtained. □

Remark 1. When $\delta = 0$, scheme (2.10)–(2.12) degenerates to the Euler fully discrete discontinuous Galerkin scheme of [10]. From the analysis, we can note that the method demonstrated here is still valid for $\delta = 0$.

4. Analysis for the CN DFDSD scheme

Applying the treatment analogous to that in Section 3 for the Euler DFDSD scheme (2.10)–(2.12), we can establish the theoretical analysis for the CN DFDSD scheme (2.18)–(2.20). Here, we provide only some concerned results on the stability and the error estimates.

Let $\tilde{w}^n_{\beta_n} = \beta_n \cdot \nabla\tilde{w}^n$, $n = 1, 2, \ldots, N$, and define

$$H(w^n, v; w^{n-1}) = (\Delta_t w^n + \tilde{w}^n_{\beta_n} + \sigma_n\tilde{w}^n, v + \delta v_{\beta_n})_\Omega + \sum_{k\in\mathcal{T}^n_h}\langle\tilde{\sigma}_n[\tilde{w}^n], v_+\rangle_{\partial k_-}. \tag{4.1}$$

Obviously, the CN DFDSD scheme (2.18) can be written as:

$$H(U^n, v; U^{n-1}) = (f_n, v + \delta v_{\beta_n}), \quad \forall v \in V_h. \tag{4.2}$$

Similarly to Lemma 3.2, we have the following lemma.

Lemma 4.1. *There exist constants $C_0, C_1 > 0$, such that, for arbitrary $w^n, w^{n-1} \in V_h$, and $w^n_-|_{\Gamma_-} \in L^2(\Gamma_-)$,*

$$H(w^n, \tilde{w}^n; w^{n-1}) + C_0\|\tilde{w}^n\|^2 + \|w^n\|^2 + \|w^{n-1}\|^2 + \frac{\sigma_1}{2}|\tilde{w}^n_-|^2_{\Gamma_-}$$
$$\ge \frac{1}{2}\left[\Delta_t\|w^n\|^2 + \sigma_0|[\tilde{w}^n]|^2_{Q^n_-} + \sigma_0|\tilde{w}^2_{\Gamma_+}\right] + C_1\delta\|\tilde{w}^n_{\beta_n}\|^2. \tag{4.3}$$

Proof. Taking $v = \tilde{w}^n$ in $H(w^n, v; w^{n-1})$ and noting that

$$(\Delta_t w^n, \tilde{w}^n) = \frac{1}{2}\Delta_t \|w^n\|^2,$$

$$|\Delta_t w^n, \delta \tilde{w}^n_{\beta_n}| \le \|w^n\|^2 + \|w^{n-1}\|^2 + \frac{\delta^2}{4\tau^2}\|\tilde{w}^n_{\beta_n}\|^2,$$

the estimates for other terms are similar to Lemma 3.2, and we can easily derive (4.3). □

By Lemma 4.1, and using a similar argument to prove Theorem 3.1 in Section 3, we can obtain the stability estimate for the CN DFDSD scheme.

Theorem 4.1. *For $\Delta t\ (= \tau)$ small enough, the CN DFDSD scheme (2.18)–(2.20) has a unique solution $\{U^n\}$, which satisfies the following stability estimate:*

$$\max_{1\le n\le N} \|U^n\|^2 + \sum_{n=1}^{N}(|[\tilde{U}^n]|^2_{Q^n_-} + |\tilde{U}^n_-|^2)\tau + \delta \sum_{n=1}^{N} \|\tilde{U}^n_{\beta_n}\|^2\tau \le C(\|f\|^2_{L^2(L^2(\Omega))} + \|g\|^2_{L^2(L^2(\Gamma_-))} + \|u_0\|^2), \tag{4.4}$$

where $C > 0$ is a constant independent of τ, h.

Similarly to Section 3, we define $\tilde{u}$, ξ, η. To get error estimates, let us prove the following lemma first.

Lemma 4.2. *There exists a constant $C > 0$, such that*

$$\begin{aligned}&\Delta_t\|\xi\|^2 + |[\tilde{\xi}^n]|^2_{Q^n_-} + |\tilde{\xi}^n_-|^2_{\Gamma_+} + \delta\|\tilde{\xi}^n_{\beta_n}\|^2\\ &\quad\le C\{\|\xi^n\|^2 + \|\xi^{n-1}\|^2 + \|\eta^n\|^2 + \|\eta^{n-1}\|^2 + h\|\tilde{\eta}^n\|_1^2 + |\tilde{\eta}^n|^2_{Q^n_-} + |\tilde{\eta}^n|^2_{\Gamma^n_+} + \|E_2^n\|^2\}.\end{aligned} \tag{4.5}$$

Proof. Following the step of proving Lemma 3.3, here we only need to estimate the term $(\Delta_t\eta^n, \delta\tilde{\xi}^n_{\beta_n})$. In fact

$$(\Delta_t\eta^n, \delta\tilde{\xi}^n_{\beta_n}) \le C(\|\eta^n\|^2 + \|\eta^{n-1}\|^2) + \varepsilon\frac{\delta^2}{4\tau^2}\|\tilde{\xi}^n_{\beta_n}\|^2.$$

Therefore, when selecting ε properly and τ small enough, then (4.5) holds.

Let u be the solution of (2.1)–(2.3). Suppose

$$u \in L^\infty(H^{r+1}(\Omega)) \cap C(\bar{\Omega}\times[0,T]), \qquad \frac{\partial u}{\partial t} \in L^2(H^{r+1}(\Omega)),$$

$$\frac{\partial^2 u}{\partial t^2} \in L^2(H^1(\Omega)), \qquad \frac{\partial^3 u}{\partial t^3} \in L^2(L^2(\Omega)).$$

Now we estimate $\tau\sum_{n=1}^N \|E_2^n\|^2$. In fact

$$\tau\sum_{n=1}^N \left\|\Delta_t u^n - \left(\frac{\partial u}{\partial t}\right)_n\right\|^2 \le Cr^4 \left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{L^2(L^2(\Omega))},$$

$$\tau\sum_{n=1}^N \|\beta_n\cdot\nabla(\tilde{u}^n - u^n) + \sigma_n(\tilde{u}^n - u^n)\|^2 \le \left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L^2(H^1(\Omega))}.$$

Hence, we can easily get

$$\tau\sum_{n=1}^N \|E_2^n\|^2 \le \left(\left\|\frac{\partial^2 u}{\partial t^2}\right\|^2_{L^2(H^1(\Omega))} + \left\|\frac{\partial^3 u}{\partial t^3}\right\|^2_{L^2(L^2(\Omega))}\right). \tag{4.7}$$

Then, we can obtain error estimates for the CN DFDSD scheme. □

Theorem 4.2. *Let u, $\{U^n\}_0^N$ be the solution of (2.1)–(2.3) and (2.18)–(2.20), respectively, and (4.5) hold. Then there exists a constant $C > 0$ independent of τ, h, such that for τ small enough,*

$$\max_{0\le n\le N} \|e^n\|^2 + \sum_{n=1}^{N}(|[\tilde{e}^n]|^2_{Q^n_-} + |\tilde{e}^n_-|^2_{\Gamma_+})\tau + \delta\sum_{n=1}^{N}\|\tilde{e}^n_{\beta_n}\|^2\tau \le C(h^{2r+1} + \tau^4), \tag{4.8}$$

where $U^n_+|_{\Gamma_+} = u^n_+|_{\Gamma_+} = \tilde{u}^n_+|_{\Gamma_+} = 0$.

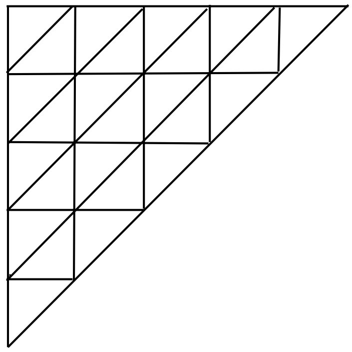

Fig. 1. Domain Ω.

Table 1
Computational results of the Euler DFDSD and CN DFDSD schemes.

Nodes (x, y)	$h = \tau = 0.05,\ \delta = 0.01$			$h = \tau = 0.005,\ \delta = 0.01$		
	Solution	Euler DFDSD	CN DFDSD	Solution	Euler DFDSD	CN DFDSD
(0.050, 0.100)	1.0000	1.0335	1.0247	1.0000	0.9956	0.9973
(0.050, 0.150)	1.0000	0.9673	0.9763	1.0000	0.9950	0.9954
(0.050, 0.200)	1.0000	0.9558	0.9655	1.0000	0.9950	0.9953
(0.050, 0.250)	1.0000	0.9572	0.9673	1.0000	0.9950	0.9953
(0.050, 0.300)	1.0000	0.9559	0.9646	1.0000	0.9950	0.9953

Table 2
Computational comparison of DG, SD and DSD schemes.

Nodes (x, y)	$h = \tau = 0.05,\ \delta = 0.01$			$h = \tau = 0.005,\ \delta = 0.01$		
	DG scheme	SD scheme	DSD scheme	DG scheme	SD scheme	DSD scheme
(0.050, 0.100)	1.0475	1.0367	1.0254	1.0471	0.9954	0.9976
(0.050, 0.150)	1.0023	0.9645	0.9736	1.0027	0.9932	0.9949
(0.050, 0.200)	0.9865	0.9737	0.9656	0.9859	0.9954	0.9957
(0.050, 0.250)	0.9604	0.9562	0.9674	0.9650	0.9754	0.9958
(0.050, 0.300)	0.9712	0.9547	0.9648	0.9672	0.9775	0.9952

5. A numerical example

Let $u(x, y, t) = (1-\exp(-(1-x-y)t/\varepsilon))/(1-\exp(-1/\varepsilon))$ be the accurate solution of the two-dimensional hyperbolic problem

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = f, \quad (x, y, t) \in \Omega \times (0, 2], \\ &u(x, y, t)|_{\Gamma_-} = g(x, y, t), \quad t \in [0, 2], \\ &u(x, y, 0) = u_0(x, y), \quad (x, y) \in \bar{\Omega}, \end{aligned}$$

where Ω is a triangle domain encircled by three straight lines, $x = 0$, $y = 1$ and $y = x$ (Fig. 1).

According to the former definition, the inflow boundary is obviously $x = 0$ $(0 \le y \le 1)$. The values of f, u_0, g can be calculated by the given function $u(x, y, t)$. For this domain, construct a uniformly isosceles right-angled triangle mesh partition (every side of the element being parallel to the boundary of the domain). By using the DG, SD, DSD, Euler DFDSD and CN DFDSD schemes, we get some numerical results at time $t = 1.05$ as presented in Tables 1 and 2 (take $\varepsilon = 0.001$).

The numerical result shows that the DFDSD scheme holds the strongpoint of the DG and SD methods, and is more easy to program than the DSD method. When the solution $u(x, y, t)$ of the continuous problem is an oscillation function or has a local large gradient (see the example in [9]), the standard Galerkin finite element method will present serious divergent oscillation. The DG method also exhibits serious distortion. However, the DFDSD method also presents good precision just as the DSD method.

6. Conclusions

In view of our analysis, the DFDSD method has the benefit of the upwinding schemes and the simple implementation of such finite difference schemes. This method has significance in solving problems as an extension to the discontinuous Galerkin methods. The stability and the error estimates for the schemes are considered in the L^2 norm in this article. We think that further theoretical work is required to build some other norm (such as the L^∞ norm) estimates for the schemes and to apply this method to more general problems.

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